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LETTER TO THE EDITOR

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Abstract
We consider a strictly isolated single-mode optical cavity resonating at angular
frequency, ω, containing atoms whose one-electron level energies are supposed
to be $\hbar\omega, 2\hbar\omega \dots B\hbar\omega$, and m photons. If the atoms are initially in their highest
energy state and $m = 0$, we find that at equilibrium: variance(m)/mean$(m) =
(B + 1)/6$, indicating that the internal field statistics is sub-Poissonian if the
number of atomic levels B does not exceed 4. Remarkably, this result does not
depend on the number of atoms, nor on the number of electrons that each atom
incorporates. Our result has application to the statistics of the light emitted by
pulsed lasers and nuclear magnetic resonance. On the mathematical side, the
result is based on the restricted partitions of integers.

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1. Introduction

We consider a single-mode optical cavity containing identical atoms. The number of photons
in the cavity, denoted by m, may be measured at any time t: It suffices in principle to quickly
remove the atoms at that time and introduce an ideal light detector. The photo-count tells us how
many photons were present in the cavity at the time considered. By performing measurements
on a large number of similarly prepared cavities, the probability $P(m, t)$ that some m-value be
found at time t is evaluated. We are particularly interested in the so-called Fano factor, defined
for example in [1]: $F(t) \equiv$ variance(m)/mean(m). In the course of time, the system eventually
reaches a state of equilibrium, in which case $P(m, t)$ and $F(t)$ are time-independent.

If the system is in a state of *thermal* equilibrium, the photon statistics is, as is well known,
that of Bose–Einstein. In that case, the Fano factor is equal to mean$(m) + 1$. This situation
occurs when the cavity may exchange energy with a thermal bath (canonical ensemble). This
would also be the case for the strictly isolated cavities (micro-canonical ensemble) considered
in the present paper if the response of the atoms to the field were *linear*. But in general

the atomic response is nonlinear and the photon distribution does not follow Bose–Einstein statistics.

This may be seen from a simple example. Consider an isolated single-mode cavity resonating at angular frequency ω and containing two identical (but distinguishable) resonant two-level atoms. The atomic ground-state energy is taken as equal to 0 and the upper-level energy as equal to $\hbar\omega = 1$, for brevity. If, initially, the two atoms are in their upper state and the cavity does not contain any photons, i.e., $m(0) = 0$, the total matter + field energy $U = 2$. Part of the atomic energy gets converted into photons in the course of time. The fundamental law of statistical mechanics tells us that, once a state of equilibrium has been reached, all the microstates of the isolated system presently considered are equally likely to occur. The complete list of microstates (first-atom state, second-atom state, number of photons) reads: (1, 1, 0), (0, 1, 1), (1, 0, 1) and (0, 0, 2). It follows from this list that the probabilities of having zero, one and two photons are proportional to 1, 2 and 1, respectively. This is obviously a non-thermal distribution. In that example, the Fano factor defined above is $F = 1/2$. But the mean value of m is equal to 1, so that the Bose–Einstein distribution would instead give $F = 2$. Sets of microstates may be obtained similarly for total-energy values $U = 0, 1, 2, 3 \ldots$. When the cavity is in contact with a bath at temperature T, the probability that the total energy be U is given by the Boltzmann probability law $\exp(-U/T)$. The end result for the photon statistics is of course Bose–Einstein. Only isolated cavities with a particular value of the total energy U are presently considered. It is straightforward to generalize the two two-level atoms result to any number M of distinguishable two-level atoms. We find that F remains equal to 1/2. This is a special case ($B = 2$) of the general result to be derived in the present paper. The Bose–Einstein statistics would, in that case, give $F = 1 + M/2$.

The Fano factor has been evaluated in many papers dealing with laser light, for example [1, 2]. In lasers, the atoms are driven to their excited state by a pump, and the cavity suffers from some optical loss, perhaps as a result of light transmission through partially reflecting mirrors. It could be thought at first that the Fano factor of isolated cavities obtains from the laser theory result by letting the pumping rate J as well as the optical loss α go to zero. This is not the case, however, because in the laser system the total (atom + field) energy U in the cavity may drift slowly, no matter how small J and α are. It follows that the variance of m deduced from laser theories in that limit does not coincide with the present statistical mechanical result, applicable to strictly isolated systems, even if the average value of U is the same in both cases. This point has been discussed in detail in [3]. Further conceptual details can be found in [4]. The present theory is nevertheless of practical significance. It is applicable to *pulsed* rather than continuous electromagnetic generators, as we later discuss.

To summarize, if initially $m = 0$ and the piece of matter is in its highest energy state, m at some later time represents the energy subtracted from the piece of matter. The probability that some m value occurs is proportional to the matter statistical weight (number of distinguishable configurations) W_m according to the equal-weight principle of statistical mechanics. We therefore need only consider the statistical weight of the atomic collection. We will consider first the case of a single atom with level energies $1, 2, \ldots, B$, and subsequently any number, M, of B-level atoms. Finally, we consider for the sake of illustration a radio-frequency cavity containing M bismuth nuclei (spin 9/2) immersed in a magnetic field. Our general result gives in that case a Fano factor equal to 11/6.

Our simple and general expression for the Fano factor derives from a property of the number of restricted partitions of integers.

2. One B-level atom

Consider an atom whose one-electron level energies are $1, 2, \ldots, B$, with $N \leqslant B$ single-spin electrons. According to the Pauli exclusion principle each level may be occupied by zero or one electrons. The atom energy is greatest when the N electrons occupy the upper states. For some subtracted energy m the statistical weight W_m is the number $p(N, m)$ of partitions of m into at most N parts, none of which exceed $B - N$. This conclusion is reached by shifting electrons downward beginning with the bottom one until the specified energy m is subtracted. Let us recall that a partition of m is a non-increasing sequence of positive integers summing up to m. By convention the number of partitions of 0 is 1. It is known [5] that

$$g(q) \equiv \sum_{m \geqslant 0} p(N, m) q^m = \prod_{i=1}^{B-N} \frac{1 - q^{N+i}}{1 - q^i}. \tag{1}$$

If the moments of m with respect to the statistical weight W_m are defined as

$$\overline{m^n} = \frac{\sum_{m=0,1,2,\ldots} m^n p(N, m)}{\sum_{m=0,1,2,\ldots} p(N, m)} \tag{2}$$

we have, as is well known [6]

$$\text{mean}(m) = \overline{m} = h'(1) \tag{3}$$

$$\text{variance}(m) = \overline{m^2} - \overline{m}^2 = h''(1) + h'(1) \tag{4}$$

where $h(q) = \ln[g(q)]$ and primes (double primes) denote first (second) derivatives with respect to the argument.

Since

$$\ln\left(\frac{1 - q^{N+i}}{1 - q^i}\right) = \ln\left(1 + \frac{N}{i}\right) + \frac{N}{2}(q - 1) + \frac{N}{24}(N + 2i - 6)(q - 1)^2 + \text{O}((q - 1)^3) \tag{5}$$

we obtain after summation over i from 1 to $B - N$

$$\text{mean}(m) = h'(1) = \tfrac{1}{2} N(B - N) \tag{6}$$

$$\text{variance}(m) = h''(1) + h'(1) = \tfrac{1}{12} N(B - N)(B + 1) \tag{7}$$

$$\frac{\text{variance}(m)}{\text{mean}(m)} = \frac{B + 1}{6}. \tag{8}$$

For example, if $B = 4$, $N = 2$, direct examination shows that $p(2, m) = 1$ if $m = 0, 1, 3, 4$ and $p(2, 2) = 2$. Therefore, mean$(m) = 2$ and variance$(m) = 10/6$ in agreement with (8). If the equilibrium field is allowed to radiate into free space the statistics of the emitted photons is sub-Poissonian (variance less than the mean) when $B < 5$ and super-Poissonian (variance exceeding the mean) when $B > 5$. The result in (8) was given in [4] without a proof.

3. Any number of B-level atoms

Let the cavity now contain a collection of M atoms labelled by $k = 1, 2, \ldots, M$, with $N^{(k)} \leqslant B$ electrons in atom k. These atoms are supposed to be distinguishable and to be coupled to one another only through energy exchanges with the cavity field.

The photon number m represents the energy subtracted from the atomic collection. The atomic statistical weight W_m is the sum, for all values of $m_1, m_2, \ldots$ summing up to m, of the products of the individual statistical weights defined above:

$$W_m = \sum_{m_1+m_2+\cdots+m_M=m} p(N^{(1)}, m_1) p(N^{(2)}, m_2) \ldots p(N^{(M)}, m_M). \tag{9}$$

The moments of m may therefore be calculated as

$$\overline{m^n} = \frac{\sum (m_1 + m_2 + \cdots + m_M)^n \, p(N^{(1)}, m_1) \, p(N^{(2)}, m_2) \ldots p(N^{(M)}, m_M)}{\sum p(N^{(1)}, m_1) \, p(N^{(2)}, m_2) \ldots p(N^{(M)}, m_M)} \tag{10}$$

where the summation is over all non-negative values of $m_1, m_2, \ldots, m_M$. It follows that the mean value of m is the sum of the individual atoms means, and that the variance of m is the sum of the individual atoms variances.

For the B-level atoms considered here, the result (6) gives

$$\text{mean}(m) = \sum_{k=1}^{M} \text{mean}^{(k)} = \sum_{k=1}^{M} \frac{N^{(k)}(B - N^{(k)})}{2} \tag{11}$$

and from (7)

$$\text{variance}(m) = \sum_{k=1}^{M} \text{variance}^{(k)} = \sum_{k=1}^{M} \frac{N^{(k)}(B - N^{(k)})(B + 1)}{12}. \tag{12}$$

Therefore, the simple result in (8) holds for any collection of B-level atoms.

4. Application to nuclear magnetic resonance

As is well known, a spin $1/2$ charged particle such as an electron immersed in a magnetic field behaves in the same manner as a (one-electron) two-level atom. This analogy generalizes to spin-s particles. We may therefore consider, as an example of the application of the previous expressions, an electro-magnetic cavity containing M identical spin-s charged particles. These particles may be distinguished from one another by their locations. If these particles are submitted to a magnetic field of appropriate strength, and in appropriate energy units, the energy levels are $-s, -s + 1, \ldots, s$. In a cold environment only the lowest energy levels are populated. But it suffices to apply the so-called π-radio-frequency pulse to get the highest levels populated. Our previous result: $\text{variance}(m)/\text{mean}(m) = (B+1)/6 = (s+1)/3$ applies once a state of equilibrium between the particles and the field has been reached. It is here supposed that the nuclei natural relaxation time is much longer than the time required for the particle-field equilibrium to be attained. If the field is allowed to radiate into free space, the emitted electro-magnetic pulse is Poissonian ($\text{variance}(m) = \text{mean}(m)$) for spin-2 particles.

Bismuth nuclei were found by Black and Goudsmit in 1927 to have a maximum spin $s = 9/2$ [7]. It follows that in the presence of a magnetic field these nuclei exhibit $B = 2s+1 = 10$ evenly spaced energy levels. When located in a radio-frequency cavity (whose resonating frequency should be in the 100 MHz range for usual magnetic-field strengths), the equilibrium Fano factor reads according to our theory $F = (B + 1)/6 = 11/6$. Due to their small energy, radio-frequency photons may be counted only at very low temperatures. It is also at such low temperatures that long nuclei relaxation times may occur.

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